Two-dimensional nonlinear diffusive numerical simulation of geomorphic modifications to cinder cones

Mattia de’Michieli Vitturi1* and J Ramón Arrowsmith2

1 Istituto Nazionale di Geofisica e Vulcanologia, Sezione di Pisa, Pisa, Italy
2 School of Earth and Space Exploration, Arizona State University, Tempe, AZ, USA

ABSTRACT: The temporal evolution of simple landforms such as cinder cones by nonlinear diffusive processes is studied through the use of a new 2D numerical model using well-established and accurate numerical mathematics and high-resolution digital elevation models (DEMs). Extending 1D (profile) nonlinear diffusion analyses used in cinder cone, hillslope and fault scarp evolution studies, we have implemented a 2D numerical model with a spatially and temporally varying sediment transport rate coefficient scaled nonlinearly by the ratio of local slope to critical slope. The high accuracy and efficient numerical implementation are documented in the paper and the MATLAB toolkit developed is used to solve for the development of an initial 2D cone form. First, we examine the nonlinear transport rule and suggest a refinement that accounts explicitly for flux at threshold slopes. We find that the maximum diffusion (necessarily introduced in the numerical model to avoid infinite rates) at the critical slope controls the final morphology, especially approaching steady state. Secondly, solving the landscape evolution problem in 2D enables a natural accounting for sediment flux convergence or divergence in the profile. Thirdly, the boundary behavior of a given landscape element controls much of what happens in that domain and so we allow for arbitrary flux magnitude or elevation boundary conditions. Fourthly, landscapes are heterogeneous in their surface cover and so we allow for spatially and temporally varying transport rate and we permit an arbitrary vertical displacement field within the model domain. To test the new formulation for the nonlinear term, the effect of variable diffusivity k and the numerical schemes implemented, we apply the model to cinder cones built on the flanks of Mount Etna in 2001 and 2002–2003. We explore the effects of DEM resolution with data from the 2001 cone and the utility of spatially variable diffusivity k to explain the variation in erosion measured by differencing repeat light detection and ranging (LiDAR) surveys gathered in 2004 and 2007 over the 2002–2003 cone complex. Copyright © 2013 John Wiley & Sons, Ltd.

KEYWORDS: nonlinear diffusion; digital elevation models (DEMs); numerical modeling; cinder cones

Introduction

Application of a generalized transport law to the temporal evolution of simple landforms has a landscape development domain over a landscape for which fluvial or landsliding processes are not important that it is regarded as a function of the geometrical \( h \) and temporal variables \( t \), and the sediment flux is assumed to take place at a rate proportional to the gradient of the elevation, analogously to the theory of heat conduction (Fourier, 1822), in which heat flux is proportional to thermal gradient. The material flux \( q_s \) can be expressed as

\[ q_s = -kh \]

where \( k \) \( [L^2T^{-1}] \) is an empirical constant coefficient representing the ‘mobility of the surface cover’ (Culling, 1960). As a consequence, if we consider also a subsidence/uplift term \( C_0 = C_0(x,y) \) \( [LT^{-1}] \) (Roering et al., 1999, 2001; Perron, 2011) function of the geometrical variables, the continuity equation is written as

\[ \rho_s \frac{\partial h}{\partial t} = \rho_s \nabla q_s + \rho_s C_0 = -\rho_s k \nabla^2 h + \rho_s C_0 \quad (1) \]

where \( \rho_s \) and \( \rho_r \) are the bulk densities of rock and sediments, respectively \([ML^{-3}]\).
Nash (1980) and Hanks et al. (1984) showed that scarp-like forms (such as fault scarps, fluvial terraces, shorelines and marine terraces) were landforms that carried accessible information about landscape history and that could be studied using a one-dimensional analytical linear diffusion approach (sediment flux linear with topographic slope). Numerous subsequent analyses examined the development of such landforms (reviewed in Hanks, 2000) in a range of climate settings. Applications of the diffusion model to fault scarps and shorelines found that in some cases the observed profiles were fit better by models in which the sediment flux increased nonlinearly with slope (Andrews and Bucknam, 1987), or by non-local models as an alternative to nonlinear diffusion models (Fouxoula-Georgiou et al., 2010). In many cases, the linear diffusion model does not sufficiently account for the increase in sediment flux as hilltops steepen (e.g. Andrews and Bucknam, 1987; Hooper and Sheridan, 1998; Roering et al., 1999; Mattson and Bruhn, 2001; Pelletier and Cline, 2007).

Hooper and Sheridan (1998) and Pelletier and Cline (2007) discuss the evolution of cinder cones by slope-dependent processes. They explored changing morphometry of cone form and showed that, in most cases, the relatively straight steep slope segments that connect the rounded crests and bases of the cones are not well explained by a linear dependence of sediment flux on topographic slope. Instead, Hooper and Sheridan (1998) use a power function of slope length and slope (exponent >1 for the latter) to simulate slope wash processes (e.g. Carson and Kirkby, 1972). Pelletier and Cline (2007) show that the evolution of cinder cones can be explained with a sediment flux rule nonlinearly related to local slope only.

In this paper, we build on the ideas of Hanks et al. (1984) and Andrews and Bucknam (1987) and the numerous papers that followed by using well-established and accurate numerical mathematics to assemble and document a flexible toolkit for exploring transport-limited nonlinear diffusion in two spatial dimensions. We follow the application of the model as demonstrated by Pelletier and Cline (2007) to cinder cone evolution. First, we examine the nonlinear transport rule and suggest a refinement that accounts explicitly for flux at threshold slopes. Secondly, solving the landscape evolution problem in 2D enables a natural accounting for sediment flux convergence or divergence in the profile. Thirdly, the boundary behavior of a given landscape element controls much of what happens in that domain (e.g. Armstrong, 1987) and so we allow for arbitrary flux magnitude or elevation boundary conditions. Fourthly, landscapes are heterogeneous in their surface cover and so we allow for spatially and temporally varying transport rate $k$ and we permit an arbitrary vertical displacement field within the model domain through the term $C_0$. The documentation of our high-accuracy and efficient numerical implementation and the available MATLAB code should be a useful and extensible contribution to landscape evolution modeling, in particular applied to high-resolution topographic datasets.

To test the new formulation for the nonlinear term, the effect of variable diffusivity $k$ and the numerical schemes implemented, the model is applied here to the erosion of scoria cones. The progressive change of morphometric parameters with increasing age is the basis for morphological dating of cones, not only on Earth (e.g. Wood, 1980a, 1980b; Inbar et al., 2011) but also for various comical structures seen on images of the Moon, Mars and Venus. In our work, we focus our attention on a simple initial topographic cone and finally we apply the proposed model to a natural example of repeat high-resolution topographic survey over cinder cones at Mount Etna, Italy, to assess sediment flux scaled by insolation or wind direction.

**Modification of the Nonlinear Flux Rule for Diffusion Erosion**

Taking into account the nonlinear increase in sediment flux as a critical slope ($S_c$) is approached, the flux can be defined as

$$q_i = f(h)\mathbf{\nabla}h$$  

(2)

where $f(h) \left[ L^2 T^{-1} \right]$ is a function of the diffusivity constant $k$ and the elevation $h$ (or generally of the slope $|\mathbf{\nabla}h|$), defining the degree of nonlinearity of the transport process. In Pelletier and Cline (2007) for example, the nonlinear term $f(h)$ is defined as

$$f(h) = \frac{k}{1 - \left(\frac{h}{S_c}\right)^n}$$  

(3)

where $S_c$ is the tangent of the angle of stability and $n$ is an empirical exponent (typically equal to 2; Andrews and Bucknam, 1987; Roering et al., 1999; Pelletier and Cline, 2007).

The nonlinear formulation defined by Equation (3) is generally accepted (e.g. Andrews and Bucknam, 1987; Roering et al., 1999; Mattson and Bruhn, 2001; Pelletier and Cline, 2007; Perron, 2011). The formulation approaches the linear rule at small slope and the fluxes go to infinity as the critical slope is approached. Andrews and Bucknam (1987) and Roering et al. (1999) suggested that the infinite fluxes at $S_c$ were physically reasonable by assuming a sliding or frictional threshold for particles on a slope. Roering et al. (1999) also made a useful distinction between the critical hillslope gradient $S_c$ (which would not be observable in the field because such slopes would rapidly decline due to high sediment flux) and a threshold hillslope gradient $S_c$ (something like the angle of repose and the maximum observed slope over slope for which our assumption of cohesionless materials holds).

While the nonlinear formulation provides a valuable basis for exploring hillslope evolution, its solution analytically is limited to strict assumptions which are more easily satisfied in one dimension. For a flexible capability, a numerical solution is required but that is challenged in particular by the nonlinear assumption (e.g. Perron, 2011). When the uplift source term is equal to zero and the initial conditions contain no slopes, the formulation approaches the linear rule at small slope and the fluxes go to infinity as the critical slope is approached. Andrews and Bucknam (1987) and Roering et al. (1999) suggested that the infinite fluxes at $S_c$ were physically reasonable by assuming a sliding or frictional threshold for particles on a slope. Roering et al. (1999) also made a useful distinction between the critical hillslope gradient $S_c$ (which would not be observable in the field because such slopes would rapidly decline due to high sediment flux) and a threshold hillslope gradient $S_c$ (something like the angle of repose and the maximum observed slope over slope for which our assumption of cohesionless materials holds).

In some complex explorations of landscape evolution, either at the beginning (Figure 1) or during the modeling (Figure 2), slopes may exist or be created which exceed the critical slope defined by $S_c$. It is often convenient to run the model with its geomorphic transport rule over those slope segments, and the rate at which they change depends on the maximum flux that is specified. We present here two examples to better understand the effects of the threshold $c_{max}$ on the solutions of the nonlinear diffusion model for those cases in which the current slope can exceed $S_c$. 

The first example is represented by a scarp where the initial topography at time $t=0$ has a maximum slope larger than the $S_c$ fixed for this test at 60° (Figure 1, black solid line in A). Due to the infinite flux, an analytical solution of the diffusion model should change the topography instantaneously in order to have everywhere a slope below the critical angle, represented in Figure 1 by the thick dashed line. When $c_{\text{max}}$ is fixed in the numerical code, as prescribed by Equation (4), it takes some time (depending on the magnitude of $c_{\text{max}}$) to reach $S_c$ (Figure 1). Thus, even if the final steady solution is the same for every $c_{\text{max}}$, the transient evolution of the solution will be different and depend on the value of $c_{\text{max}}$ (see the profiles at time $t=1$ ka in Figure 1), because a higher maximum flux threshold yields the quickest slope decrease. Wallace (1977) argued that these early discrete failures on initially steep fault scarps would adjust the slope to below critical in ~100 years based on his observations from the Basin and Range – a small fraction of the age of the landform and also by processes not represented by the nonlinear transport law. Nevertheless, we argue that in some cases nonlinear diffusive effects can be important in these early times. Explicit specification of the $c_{\text{max}}$ with knowledge of the implied response time provides additional opportunities for quantifying process.

Another example highlighting the role of the threshold $c_{\text{max}}$ is presented in Figure 2 where, for an initial flat topography with a constant uplift applied to the right section, three different values of $c_{\text{max}}$ are tested (50, 100 and 1000 m² ka⁻¹). For all the runs at the beginning the slope created by the uplift is small, the nonlinear effects are negligible and the maximum slope increases in time almost linearly. After about 1.5 ka, the nonlinear diffusion increases, becoming a limiting factor for further slope steepening. For the smallest value of $c_{\text{max}}$ the maximum flux is reached at about 8.5 ka and after that the maximum slope starts to increase at a faster rate. The same thing occurs for $c_{\text{max}}=100$ m² ka⁻¹ after ~17.5 ka, and eventually for the maximum value of $c_{\text{max}}$. For larger values of $k$ and/or $c_{\text{max}}$, or for smaller values of the uplift rate $C_0$, the critical slope could not be reached in a finite time, and the final solution is strongly dependent on the ratio between $C_0$ and $c_{\text{max}}$.

When a numerical threshold is introduced into the model, as done with Equation (4), the maximum value of the flux is not reached at the critical slope, but at a smaller angle totally dependent on the ad hoc threshold, and this angle is a function of $c_{\text{max}}$. This is clearly seen in the transition in the erosional pattern presented in the zoomed plot of Figure 2 for $c_{\text{max}}=50$ m² ka⁻¹, when the maximum slope starts to increase at a faster rate before reaching the critical angle of 33°. Furthermore, if $\nabla^2 h > S_c$, the nonlinear term $f(h)$ in the original formulation given by Equation (3) becomes negative, creating antidiffusion that amplifies small perturbations in elevation (Perron, 2011). These facts, together with the examples presented above, encouraged us to modify the original formulation of the nonlinear model in the following way:

$$ f(h) = \frac{k}{1 - \left(\frac{c_{\text{max}}}{c_{\text{max}}} - 1\right) \min \left(\frac{1}{\nabla^2 h} \frac{1}{S_c}\right)} $$  \hspace{1cm} (5)

Thus, as suggested by Pelletier (2008), a threshold on the maximum flux is fixed, but with the advantage that the explicitly specified maximum flux $c_{\text{max}}$ is attained exactly at the critical slope. Furthermore, as happens for Equation (4), when $c_{\text{max}} \to +\infty$ the modified nonlinear term defined by Equation (5) converges to the original formulation given by Equation (3), thus satisfying the commonly desired linear behavior at low slopes and infinite
flux at \( S_r \). Additionally, the introduction of the function \( \min (1; | \nabla h | S_r) \) in the denominator of Equation (5) avoids negative values of the nonlinear term and thus antidiffusion problems. The effects of the threshold \( c_{\max} \) on the new function \( f \) defined by Equation (5) and the relative error with respect to the original formulation are presented in Figure 3 for \( k = 1 \text{ m}^2 \text{ ka}^{-1} \).

Furthermore, differently from Pelletier and Cline (2007) and most other related studies, we allow the diffusivity constant \( k \) to be a function of \( (x, y, t) \) and \( h \), and thus to vary with space and time or with the solution, for example to consider the different exposure to the wind or the sun, or to consider that in cinder cones the peaks of erosion generally occur in the first stages after the eruption, with the stripping of fine ash (Inbar et al., 2011). With this assumption, the divergence operator applied to the flux \( q \) in Equation (1) acts also on the coefficient \( k \) and, when substituting Equation (5) in Equation (2) and then Equation (2) in Equation (1), we obtain the following nonlinear diffusion model for sediment transport:

\[
\frac{\partial h}{\partial t} = \rho_s \nabla \cdot \left[ \frac{k}{\min \left( 1; \frac{|\nabla h| S_r}{c_{\max}} \right)} \nabla h \right] + \rho_s c_0 \quad (6)
\]

This equation predicts nonlinearly increasing sediment flux (with an explicitly specified maximum flux) as the angle of stability is approached, and reduces to the linear diffusion equation (Culling, 1960) for \( |\nabla h| S_r \ll S_r \) and to the original nonlinear diffusion equation for \( c_{\max} \rightarrow +\infty \) (cf. Andrews and Bucknam, 1987; Roering et al., 1999; Pelletier and Cline, 2007).

Numerical discretization

Analytical methods (e.g. the Fourier method) can be used for solving linear parabolic equations. However, for nonlinear problems such methods generally fail and we are restricted to numerical solutions. When a numerical discretization is applied to parabolic equations, like those considered here, the stability constraint on the time-step for explicit methods (\( \Delta t = O(dx^2) \)) can be prohibitive and implicit methods are necessary. In this section, we focus on the implementation of the alternating direction implicit (ADI) scheme for the temporal discretization of the 2D model defined by Equation (6). The main idea of ADI methods is to solve the two-dimensional problem in two steps, treating only one derivative at each step, with the advantage that the algebraic equations obtained with the numerical discretization have a simpler structure and are thus easier and faster to solve. This technique has been successfully applied to the linear hillslope transport law (Perron et al., 2003, 2008; Pelletier, 2008) coupled with an implicit Crank–Nicolson scheme for the temporal discretization (Crank and Nicolson, 1947). As discussed in Perron (2011), implicit methods are unconditionally stable and therefore allow time-steps that are comparable to the timescale of interest in the studied system. However, the implementation of an ADI method for the nonlinear hillslope transport law is not a trivial task and alternative formulations have been proposed (Fagherazzi et al., 2002; Perron, 2011).

To overcome the restrictive limits on the stability and accuracy of landscape evolution models given by explicit numerical methods used to solve nonlinear sediment transport equations, Perron (2011) adapted a first-order accurate-in-time implicit approach developed by Fagherazzi et al. (2002). This formulation approximated the right-hand side of Equation (1) with a first-order Taylor series expansion at time \( n \) (his variable name) and then approximated the divergence of the flux in a cell \((i,j)\) of the computational grid with the sum of discretized fluxes between the cell and its neighbors. Perron (2011) modified the approach to first evaluate the flux divergence analytically and then the Taylor series expansion on the finite difference of the flux divergence is performed. The implicit methods proposed, despite offering greater stability than an explicit method with a comparable accuracy (first order), do not reach the stability and the temporal accuracy (second order) of the Crank–Nicolson method.

Our approach strengthens the numerics further. Following Witelski and Bowen (2003), we implement the ADI scheme coupled with a Crank–Nicolson method for the discretization of the nonlinear transport model. For the sake of simplicity, in the following description we consider \( \rho_s = \rho_p \) to cancel the bulk densities from Equation (6). With this assumption, considering a finite difference scheme for the time discretization involving the solutions at two consecutive time-steps \( h^n \) and \( h^{n+1} \), from Equation (6) we derive
\[
\frac{h^{n+1} - h^n}{\Delta t} = \theta \nabla \cdot (f(h^n) \nabla h^{n+1}) + (1 - \theta) \nabla \cdot (f(h') \nabla h') + C_0
\]  
(7)

where the superscript denotes discrete time-steps, \(h^n\) is the solution at \(t^n = n \Delta t\) and \(0 \leq \theta \leq 1\) is a real parameter defining the temporal discretization scheme. For \(\theta = 1/2\), Equation (7) defines the second-order accurate-in-time Crank–Nicolson scheme; for all other \(0 \leq \theta \leq 1\), the scheme is first-order accurate. For \(\theta = 0\), Equation (7) yields the explicit forward Euler scheme, while \(\theta = 1\) corresponds to the unconditionally stable backward Euler method.

We observe that for any \(\theta > 0\) Equation (7) represents an implicit problem in the unknown \(h^{n+1}\), involving the inversion of a two-dimensional nonlinear spatial operator. This becomes clear when we separate the implicit and explicit terms in Equation (7) to obtain

\[
\left[ l - \frac{1}{2} \Delta t \nabla \cdot (f' \nabla (f')) \right] h^{n+1} = \left[ l - \frac{1}{2} \Delta t \nabla \cdot (f') \right] h^n + C_0 \Delta t
\]  
(8)

where on the left-hand side we have the two-dimensional spatial operator to invert, and where we introduced the notation \(l = \theta h\) and \(lI\) for identity operator \((lI)(h) = h\).

The idea of the ADI scheme is to use an approximate factorization of the 2D operator, in order to solve Equation (8) through the inversion of two simpler 1D operators. Indeed, if we define two one-dimensional operators:

\[
L_x = l - \frac{1}{2} \frac{\partial}{\partial x} \left[ f^{n+1} \frac{\partial}{\partial x} \right]
\]  
(9)

\[
L_y = l - \frac{1}{2} \frac{\partial}{\partial y} \left[ f^{n+1} \frac{\partial}{\partial y} \right]
\]  
(10)

then the operator on the left-hand side of Equation (8) acting on \(h^{n+1}\) can be rewritten as a sum of mixed-derivative terms \(\left(\frac{\partial^2 f}{\partial x \partial y}\right)\) and products of one-dimensional operators:

\[
\left[ l - \frac{1}{2} \Delta t \nabla \cdot (f' \nabla (f')) \right] = L_x L_y
\]

\[
-\frac{1}{4} \Delta t^2 \frac{\partial^2}{\partial x^2} \left[ f'^{n+1} \frac{\partial}{\partial x} \left[ f'^{n+1} \frac{\partial}{\partial y} \right] \right]
\]  
(11)

Thus we produce a pseudo-linearization of the Crank–Nicolson method:

\[
\tilde{L}_x \tilde{L}_y h^{n+1} = \left[ l - \frac{1}{2} \Delta t \nabla \cdot (f' \nabla (f')) \right] h^{n+1} + C_0 \Delta t
\]  
(12)

The presence of the term \(f^{n+1}\) on the right-hand side makes the problem implicit. To solve it, we introduce an intermediate explicit approximation of the solution at time \(\tilde{h}^{n+1}\), denoted by \(\tilde{h}^{n+1}\), and the linear operators

\[
\tilde{L}_x = l - \frac{1}{2} \frac{\partial}{\partial x} \left( f(\tilde{h}^{n+1}) \frac{\partial}{\partial x} \right)
\]  
(13)

\[
\tilde{L}_y = l - \frac{1}{2} \frac{\partial}{\partial y} \left( f(\tilde{h}^{n+1}) \frac{\partial}{\partial y} \right)
\]  
(14)

Once \(\tilde{h}^{n+1}\) is given, Equation (15) becomes an explicit linear equation for the unknown \(h^{n+1}\). Furthermore, if the approximation \(\tilde{h}^{n+1}\) converges to \(h^{n+1}\), then the solution of the problem (15) converges to the solution of the original problem (12). Thus we define an iterative scheme in order to have \(\tilde{h}^{n+1}\) converging to \(h^{n+1}\), or the difference \(h^{n+1} - \tilde{h}^{n+1}\) going to zero.

Proceeding formally, if we let \(v = h^{n+1} - \tilde{h}^{n+1}\) and subtract \(\tilde{L}_x \tilde{L}_y \tilde{h}^{n+1}\) from both sides of Equation (15), we obtain

\[
\tilde{L}_x \tilde{L}_y v = -\left( \tilde{h}^{n+1} - h^n \right) + \frac{1}{2} \Delta t \left[ q_v(h^n) - q_v(\tilde{h}^{n+1}) \right]
\]  

\[
+ \frac{1}{4} \Delta t^2 \frac{\partial^2}{\partial x^2} \left[ f(\tilde{h}^{n+1}) \frac{\partial}{\partial x} \left[ f(\tilde{h}^{n+1}) \frac{\partial}{\partial y} \right] \right] h^n + C_0 \Delta t
\]  
(16)

The ADI split equations for this Crank–Nicolson method are then

\[
(pL) = \begin{cases} 
L_x w = -\left( \tilde{h}^{n+1} - h^n \right) + \frac{1}{2} \Delta t \left[ q_v(h^n) - q_v(\tilde{h}^{n+1}) \right] \\
L_y v = w \\
\tilde{h}^{n+1} = h^{n+1} + v
\end{cases}
\]  
(17)

where \(\tilde{h}^{n+1}\) is not given a priori. We use the ADI method \((pL)\) as a linear iterative method to approximate \(h^{n+1}\), with initial estimate \(h^{n+1}_0 = h^n\). When the convergence is reached (i.e. \(v = 0\)), the solution for Equation (16) is also a solution of the original problem formulated in Equation (7).

Because Equation (8) is second-order accurate in time, without loss of accuracy we can apply the \(O(\Delta t^2)\) operator on the right-hand side of Equation (11) to \(h^n\) instead of \(h^{n+1}\) while introducing only higher-order corrections, and we obtain the following scheme:
To obtain the full numerical scheme, we must also introduce a proper treatment of the boundary conditions and a discretization of the spatial differential terms in Equation (17). In the numerical model, it is possible to select independently for any edge of the domain a Neumann (imposed flux) or Dirichlet boundary (imposed elevation) condition, and the corresponding terms are treated implicitly in the numerical solution. For the space discretization, due to the parabolic nature of the problem, we adopted a second-order centered finite difference scheme. When coupled with the Crank–Nicholson scheme for the discretization in time, our numerical formulation leads to a fully second-order method in both space and time. We remark that when ADI approaches are used to solve nonlinear systems bias can be introduced and thus, to avoid this problem, the order of the direction calculations (x or y direction first) is switched between time steps. The use of an implicit scheme avoids the rectangular pattern that can occur during the early time-step of explicit schemes implemented on Cartesian grids.

In the Perron (2011) method, a matrix operator must be recalculated at each time-step; the operators \( L_x \) and \( L_y \) are calculated at each internal iteration of the PL method, giving a higher computational cost, but with the advantage of a second-order accuracy in time. To reduce the computational cost of our approach, a step doubling technique adapts the time-step size to the accuracy of the solution. At each time-step, the discretized equations are integrated in time twice, once with a full step \( \Delta t \), then, independently, with two half-steps \( \Delta t/2 \), obtaining two different solutions \( h_i^{n+1} \) and \( h_i^{n+1} \). A Taylor expansion of the solution shows that the difference \( \Delta h = |h_i^{n+1} - h_i^{n+1}| \) between the two numerical estimates is a convenient indicator of the accuracy of the numerical solution and allows, through a well-established PI step-size control technique (Gustafsson et al., 1988), step size adjustment to keep the local truncation error per unit step bounded. This strategy, coupled with the stability and the second-order accuracy provided by the Crank–Nicholson scheme, allows us to increase by orders of magnitude the time-step when the solution is converging to a steady state and thus compensate the additional computational cost required by the ADI method with respect to the method presented in Perron (2011).

### Numerical Results

#### Overview

The first author coded in MATLAB the discretized equations derived in the last section following the approach of Witelski and Bowen (2003), together with the procedure for the time-step size adjustment. An enveloping script was developed in order to easily define specified initial and boundary conditions, spatial and temporal changes in \( k \), \( c_{\text{max}} \), spatial variation in uplift and plotting directives. These tools are available through the Volcano Modelling and Simulation gateway (VMSg, http://vmsg.pi.ingv.it), developed and maintained by the Sezione di Pisa of the Istituto Nazionale di Geofisica e Vulcanologia.

With this toolkit, we explored the model behavior and tested the numerical scheme on a 2D synthetic cinder cone. We apply the results to the simple cinder cone form following the interesting results of Wood (1980a, 1980b), Hooper and Sheridan (1998), Inbar et al. (2011) and Pelletier and Cline (2007) in particular. Like them, we assume transport-limited conditions and that only the nonlinear diffusive processes are operating. We begin with a reprise of the Pelletier and Cline (2007) exploration of the evolution of the cone with spatially constant \( k \). Having simply validated the model, we increase complexity by considering spatially heterogeneous diffusivity (modulated, for example, by slope and aspect controlled vegetation vigor). Finally, we apply the code to cinder cone evolution on the flank of Mount Etna and compare the resulting changes with differential high-resolution topography.

#### Synthetic cone

Our exemplary cinder cone has a radially symmetric geometry (following Pelletier and Cline, 2007; Figure 4). If we denote \( r = \sqrt{x^2 + y^2} \), the equation of the cone is given by

\[
h(r) = \begin{cases} 
130 & r \leq 100 \\
130 + \frac{r - 50}{50} & 50 < r \leq 100 \\
160 - \frac{r - 100}{250} & 160.0 < r \leq 100 \\
0 & r > 100 
\end{cases}
\]

The computational domain is 500 \( \times \) 500 m, discretized with a uniform grid with a step size of 5 m (10201 grid points). For all the runs presented here, the base value for \( k \) is 1 m\(^2\) ka\(^{-1}\), the maximum diffusivity coefficient is \( c_{\text{max}} = 10^2 \text{m}^2\text{ka}^{-1} \) and the solutions are presented at time \( t = 10 \text{ka} \).

To exercise the model, we compare the results obtained varying the exponent \( n \), using the values \(+\infty\), 0 and 2. The first two values reduce to the standard linear diffusion model. In Fornaciari et al. (2012), a linear cone degradation model is used to simulate scoria cone erosion and detect morphometric changes.

The solution obtained with \( n = + \infty \) is presented in Figure 5. We observe a small rounding of the rim and base, not accompanied by slope rotation of the cone flank. The slope distribution shows that at the beginning the only angles of the synthetic topography are 0° (the region outside the cone and the small flat region inside the crater), and the angle corresponding to the flanks of the cone (~32.6192°). This angle is slightly below the critical slope, fixed here at 33°, and thus it is possible also to use the original formulation for the flux given by Equation (3). At the final time, due to erosion of the rim and the deposition at the base of the cone, there are more cells with intermediate

![Figure 4](image-url)
angles between $0^\circ$ and the maximum angle. The maximum angle remains unchanged, as expected with a linear diffusion model where the slope rotation component of the cinder cone evolution is smaller than that obtained with a nonlinear model (see below).

In Figure 6, we present the solution at time $t=10$ ka obtained with $n=0$, giving a constant value $f(h) = c_{\text{max}}$, representing again a linear model but with a diffusivity ten times larger than in the previous test. The larger erosion of the rim and deposition at the base is evident from the top-left and top-right panels, where the crater has almost disappeared. This fact is evident from the erosional pattern presented in the bottom-left panel, where we can see that the rim of the crater has been eroded by ~20 m, while inside the crater more than 5 m of material has been deposited. Even with these greater changes, the slope rotation is not significant, as also illustrated by the histogram of the slope angles presented in the bottom-right panel. In fact, the maximum angle is $\sim 32^\circ$, corresponding to a rotation of $1^\circ$ from the initial solution (black bars).

The solution obtained with $n=2$ is presented in Figure 7. In this case we have a genuinely nonlinear model, where the function $f(h)$ varies nonlinearly between 1, when the slope is flat, to $c_{\text{max}} = 10$, when the critical slope (in this case $33^\circ$) is reached and the sediment flux attains its maximum. Consequently, the average erosion is smaller than the erosion in the first test (see Figure 5) and larger than that of the second test (see Figure 6). The rim erosion is $\sim 16$ m, compared to $\sim 6$ m for the run with $n=\infty$ and $\sim 20$ m for the run with $n=0$. Despite the lower diffusivity, the nonlinear run presents a larger profile rotation, as shown by the histogram of the slope angles (Figure 7 compared with Figure 6). Linear diffusion may not properly account for the slope rotation associated with the increase in sediment flux as hillslopes steepen and transport processes become more efficient. The strategy to adjust the step size for this nonlinear case made for great efficiency: for a maximum local truncation error per unit step of the order of $10^{-5}$, the time-step increases by three orders of magnitude from the beginning of the run to the end. Furthermore, the initial
core presents an angle slightly below the critical slope and thus it is possible to use also the original formulation for the flux given by Equation (3), avoiding the problem of infinite flux. A comparison between the central profile obtained for $n = 2$ with the new formulation defined by Equation (5) and with the original one is presented in Figure 8, highlighting the small error introduced modifying the flux and thus the fact that the new formulation is an adequate approximation of the original one. This error reduces with larger values of $c_{\text{max}}$.

To examine more realistic conditions such as spatially varying transport rates, we have used the same parameters of the previous test (nonlinear exponent $n = 2$ and maximum diffusivity coefficient $c_{\text{max}} = 10$), for a calculation in which the diffusivity $k$ varies with the slope magnitude and aspect. The diffusivity is proportional to the dot product of the unit vector normal to the topography and the normal vector with an azimuth angle of 90°, and consequently the diffusivity is larger on the eastern flank than on the western flank (thus crudely simulating insolation variation). The ratio between the maximum and minimum value of $k$ has been fixed to 4 and, to compare the results obtained with the variable diffusivity with those obtained with the constant diffusivity, $k$ has been linearly rescaled to give an average value of $k_{\text{avg}} = 5 \text{ m}^2 \text{ka}^{-1}$ on the domain (similar to the average value of the previous test). The erosional pattern and the initial and final west–east profile are presented in Figure 9. From the top-right panel we see that, while the erosion is similar on the eastern and western rims, the slope rotation is present only on the eastern flank, presenting also a greater deposition at its base. In this case $k$, being a function of the actual topography, is not only a function of $(x,y)$ but changes also with time. This example is representative of the different erosional patterns observed on real cinder cones, where the exposure to the Sun and the vegetation affect the mobility of the soil.

**Application to cinder cone evolution on the flank of Mount Etna**

In this section, we apply the model to cinder cones built on the flanks of Mount Etna in 2001 and 2002–2003. We explore the effects of digital elevation model (DEM) resolution with data from the 2001 cone and the utility of spatially variable diffusivity to explain the variation in erosion measured by differencing repeat LIDAR (light detection and ranging; or laser imaging detection and ranging) surveys gathered in 2004 and 2007 over the 2002–2003 cone complex. The DEMs used for this analysis include cells in which the slope exceeds the critical value and thus we are restricted to use the new formulation for the flux given by Equation (5) to avoid the problems related to infinite fluxes. In these explorations, the transport-limited assumption remains generally valid because the cones are constructed dominantly of unconsolidated tephra. The 2001 cone is officially named ‘Monte Josemaria Escrivá’, although it is more frequently referred to as ‘Laghetto cone’ (Fornaciai et al., 2010). It is a simple cone truncated by a single, nearly circular crater. The flanks of the cone, due to instability of the deposit, present a break in slope and a bimodal distribution with modes of 31° and 35°. The inner surface of the crater, probably due to the collapse of the western rim (Calvari and Pinkerton, 2004), is irregular. LIDAR data were gathered in four surveys on 16–17 September 2004 and were used to produce a 1 m resolution DEM. The raw LIDAR data were affected by systematic errors caused by both the laser scanner system and the GPS/INS unit anomalies during data acquisition (Vosselman and Maas, 2001). The correction for systematic errors of the LIDAR data is presented in Favalli et al. (2009) and the resulting DEM is shown in Figure 10, while an analysis of the accuracy and associated error in scoria cone morphometry obtained using different DEMs with different resolution is presented in Fornaciai et al. (2012).
The 1 m DEMs provide an opportunity to study the effects of high-resolution input data on the nonlinear numerical model. The results at $t = 100$ ka of a run with a critical slope of $33^\circ$, a diffusivity coefficient $k = 5 \text{ m}^2 \text{ ka}^{-1}$, a maximum diffusivity coefficient $c_{\text{max}} = 5$, a nonlinear exponent $n = 2$ and Neumann (fixed-flux) boundary conditions are shown in Figure 11. The cone has been smoothed and the crater has been almost completely filled by the sediment deposited from the rim. The map of erosion/deposition zones highlights the erosion of the eastern rim ($\sim$20 m) and the consequent deposition inside crater ($\sim$15 m). The three profiles shown in Figure 11A are plotted for the initial and final time of the simulation. The blue profile, roughly corresponding to the west-east section of the cone, shows the rotation of the cone flanks accompanying the rim erosion.

The same parameters were used with a lower-resolution DEM, obtained by interpolating the original 1 m data on a 5 m grid and the solution at $t = 100$ ka has been upscaled linearly.
to the original grid, in order to evaluate the difference between the results obtained with the original DEM and the low-resolution one. The relative error is

$$\text{err} = \frac{(\Delta h)_i - (\Delta h)_H}{(\Delta h)_H}$$  \hspace{1cm} (19)$$

where $(\Delta h)_i$ and $(\Delta h)_H$ are the differences between the final and the initial solution for the low-resolution and high-resolution DEMs respectively (Figure 12). The maximum error is mostly concentrated at the edges of the domain, corresponding to the base of the cone, and in the proximity of small topographic features, with a ring along the flanks of the cone corresponding to the region where $(\Delta h)_i$ is minimal. In particular, greater erosion is indicated with the low-resolution DEM on the northwest of the domain, while less erosion is evident in the southeastern corner, where the difference between the solutions of the two simulations exceeds 1 m. On average, the solution predicted at 100 ka obtained with the low-resolution DEM underestimates the total erosion of the cinder cone because the increase in grid size lowers the average slope angle and thus erosion.

To show the effect of a variable diffusivity $k$ on the erosion of a real cinder cone, we apply the model to the evolution of a scoria cone complex formed at Mount Etna in 2002–2003, officially named ‘Monte Barbagallo’ (Figure 13). Its complex structure is the result of alternating phreato-magmatic and magmatic (strombolian and fountaining) activity from two craters, each containing multiple vents. Repeat LiDAR surveys in 2004 and 2007 were used to produce 1 m DEMs and to map the changing morphology of the cinder cones by Fornaciai et al. (2010).

During the 2004–2007 period, the erosion and deposition patterns are not symmetrically distributed around the cones. In particular, the greatest erosion occurred on the northeastern flank of the 2002–2003 scoria cone complex. Comparison of the 2004 and 2007 DEMs shows clear morphology changes (e.g. Fornaciai et al., 2010). Notably, the southeastern flank shows incipient slope rotation. The heterogeneous erosion may be due to more efficient downslope sediment transport driven by the effects of the predominant westerly and northwesterly winds (Fornaciai et al., 2010).

The nonlinear diffusion model with spatially constant $k$ is not able to predict the 2004–2007 erosion and deposition patterns. A simple topographic exposure model Boose et al. (1994) provides useful predictions of areas protected from or exposed to damaging winds. Each point of the DEM is classified as protected or exposed, depending on whether or not it falls within the wind shadow cast by points upwind. The wind shadow is estimated by assuming that the wind bends downward as it passes over topography. At each time-step, the exposed points are detected and then the incidence angle of the wind on the exposed points is evaluated. The diffusion coefficient $k$ in the nonlinear model becomes a function of the wind exposure, and thus of both the spatial variables $(x,y)$ and the temporal variable $t$, changing the topography with time and consequently also the exposure to the wind (even if the change with time is small with respect to the changes with space). Our implementation assumes that the wind makes the nonlinear downslope motion of the scoria more efficient. A comparison between a run without the effect of wind exposure and a run considering the increased efficiency of the motion due to a westerly wind is presented in Figure 14. In the latter case, the depth of the erosion on the eastern flank of the cinder cone complex reaches 5 m, a value comparable to the observations of Fornaciai et al. (2010), coupled with a small slope rotation of the eastern flank of both the cones. We observe that the morphology changes obtained from comparison of the 2004 and 2007 DEMs (Fornaciai et al., 2010; Figure 10) show a larger erosion of the south-southeastern portion of flank of the southern cone, probably due to the variability of the wind direction, while in the numerical test a constant westerly wind has been used.
Conclusions

Using well-established and accurate numerical mathematics, we have assembled a flexible toolkit for exploring transport-limited nonlinear diffusion and applied it to both synthetic and observed cinder cones. Our simulations of the nonlinear diffusive hillslope development of a cinder cone increase in complexity until we compare them favorably with the results of differential LIDAR measures of geomorphic changes on cinder cones at Mount Etna, where heterogeneous efficiency in sediment transport may be promoted by wind exposure.

We have built on previous contributions developing the basis for and applying nonlinear diffusion (e.g. Andrews and Bucknam, 1987; Hooper and Sheridan, 1998; Roering et al., 1999; Mattson and Bruhn, 2001; Pelletier and Cline, 2007; Pelletier, 2008) to bring the specification of maximum flux via a coefficient $c_{\text{max}}$ as an explicit variable rather than assigned ad hoc for stable numerics (Equation (5)). Because our new formulation reduces to the standard nonlinear form (Equation (3)) for large $c_{\text{max}}$, it is a useful extension which not only fixes the numerical ‘pathology’ but, more importantly, provides geomorphologists with the flexibility to explore the details of the approach to critical slope $S_c$. Direct specification of finite flux at the critical slope is enabled (e.g. simulating viscous-dominated flow of hillslope materials as they begin to detach at $S_c$). Stochastic variation in process rates near the critical slope is likely and could be implemented via a random variation of $c_{\text{max}}$, for example. Another useful extension is spatial and temporal variation in $k$ to address complex process rate distributions, including those modulated by insolation or wind. These changes are implemented in a high-order modular numerical framework which we document extensively in this paper, and the MATLAB code is available at http://vmsg.pi.ingv.it. This high-order and efficient framework is a useful complement to the Perron (2011) contribution and could be the basis for future geomorphic process explorations.

Several limitations and opportunities for extensions to this approach are evident. There are clear process limitations in our assumption of only slope dependency on sediment flux rates and transport limitation. A more complete hillslope evolution model would include contributing area dependence to account for runoff-driven processes and be coupled with a soil production function so that the transport-limited assumption would be relaxed (this was implemented partially by Hooper and Sheridan, 1998, who simulated a resistant crater rim with spatially variable transport rates). Calibrated rate constants would enable the analysis to move from morphological modeling to morphological dating and permit us to explore the climate controls (past and present) on the rates and character of cinder cone evolution (e.g. Wood, 1980b). Finally, high-resolution data are becoming increasingly available for many places on Earth as well as from Mars and the Moon, and those data and their ability to measure landform geometry and surface changes will invigorate the use of tools such as the one we present in this paper.

Acknowledgements—This work has been partially supported by the Marie Curie Actions of the European Commission in the frame of the MAMMA project (FP7-PEOPLE-2009-IOF-251833) and by US National Science Foundation grant EAR-0930643 (www.openTopography.org). The authors wish to thank Alessandro Fornaciai, Massimiliano Favalli and Simone Tarquini for their data and Amanda Clarke for discussions. Very helpful critical reviews of the manuscript by Jon Pelletier and an anonymous reviewer guided us to improve the analysis and its articulation.

References


